Biometrical Letters
Vol. 46 (2009), No. 2, 113-128

Estimation and hypothesis testing in block designs with nested rows and columns

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Dedicated to Professor Tadeusz Caliński on his 80th birthday

Summary

We present methods of estimation and hypothesis testing for treatment contrasts in generally balanced block designs with nested rows and columns. The linear mixed effects model is built by randomization of all the nuisance effects, that is blocks, rows and columns. This allows for model decomposition into five sub-models according to decomposition of the random sample space into five strata. We present methods for estimation of the treatment contrasts, for testing within the strata and also various methods of combining the strata information. In this work we largely apply the results given by Caliński and Kageyama (2000, 2003).

Key words: block design with nested rows and columns, combined estimator, combined test, general balance.

1. Introduction

We consider the following mixed effects linear model for a block design with nested rows and columns (cf Łacka and Kozłowska, 2009):

$$\mathbf{y} = \mu\mathbf{1} + \mathbf{D}'\boldsymbol{\gamma} + \mathbf{D}_1'\boldsymbol{\rho} + \mathbf{D}_2'\boldsymbol{\phi} + \boldsymbol{\Delta}'\boldsymbol{\tau} + \boldsymbol{\epsilon} + \mathbf{e}, \tag{1}$$

where $\mathbf{y}$ denotes the n-dimensional vector of observations, $\boldsymbol{\gamma}$, $\boldsymbol{\rho}$ and $\boldsymbol{\phi}$ denote b_1, b_2 and b_3-dimensional vectors of random effects of blocks, rows and columns, respectively, $\boldsymbol{\tau}$ is the v-dimensional vector of fixed treatment effects and $\boldsymbol{\epsilon}$ and $\mathbf{e}$ are n-dimensional vectors of the random effects of experimental units and of technical errors. Stochasticity of the effects of blocks, rows, columns and experimental units comes from randomization of these structural elements of the experiment. We further assume that the elements of vector $\boldsymbol{\epsilon}$ are mutually independent as well as uncorrelated with the random technical errors. $\mathbf{D}'$, $\mathbf{D}_1'$, $\mathbf{D}_2'$, $\boldsymbol{\Delta}'$ denote the design matrices for blocks, rows, columns and treatments, respectively. Assuming that the observations are written in lexicographical order, we have

$$\mathbf{D}' = \mathbf{I}_{b_3} \otimes \mathbf{1}_{b_1} \otimes \mathbf{1}_{b_2}, \qquad \mathbf{D}_1' = \mathbf{I}_{b_3} \otimes \mathbf{I}_{b_1} \otimes \mathbf{1}_{b_2}, \qquad \mathbf{D}_2' = \mathbf{I}_{b_3} \otimes \mathbf{1}_{b_1} \otimes \mathbf{I}_{b_2},$$

where matrix $\mathbf{I}_x$ denotes the identity matrix of order x, $\mathbf{1}_x$ denotes the x dimensional vector of ones and the symbol $\otimes$ denotes the Kronecker product. Then, the variance-covariance matrix of the vector $\mathbf{y}$ can be written as

$$\begin{aligned} \mathrm{Cov}(\mathbf{y}) = \;& \sigma_\gamma^2 \mathbf{D}'\mathbf{Q}_{\mathbf{1}_{b_3}}\mathbf{D} + \sigma_\rho^2 \mathbf{D}_1'(\mathbf{I}_{b_3} \otimes \mathbf{Q}_{\mathbf{1}_{b_1}})\mathbf{D}_1 + \sigma_\phi^2 \mathbf{D}_2'(\mathbf{I}_{b_3} \otimes \mathbf{Q}_{\mathbf{1}_{b_2}})\mathbf{D}_2 \\ & + \sigma_\epsilon^2 (\mathbf{I}_{b_3} \otimes \mathbf{Q}_{\mathbf{1}_{b_1}} \otimes \mathbf{Q}_{\mathbf{1}_{b_2}}) + \sigma_e^2 \mathbf{I}, \end{aligned} \tag{2}$$

where $\sigma_\gamma^2, \sigma_\rho^2, \sigma_\phi^2, \sigma_\epsilon^2, \sigma_e^2$ denote the respective variances of the model random effects and $\mathbf{Q}_{\mathbf{1}_{b_x}} = \mathbf{I} - \mathbf{P}_{\mathbf{1}_{b_x}}$, where $\mathbf{P}_{\mathbf{1}_{b_x}} = \frac{1}{b_x}\mathbf{1}_{b_x}\mathbf{1}_{b_x}'$.

According to the notation introduced by Nelder (1965b) such a model represents a design of the $\mathrm{B}(b_3) \longrightarrow (\mathrm{B}(b_1) \times \mathrm{B}(b_2))$ type. Then, the variance-covariance matrix of $\mathbf{y}$ can be written as

$$\mathrm{Cov}(\mathbf{y}) = \sum_{s=0}^{4} \xi_s \mathbf{P_s}, \tag{3}$$

where the $\mathbf{P_s}$ are as follows:

$$\begin{aligned} &\mathbf{P_0} = \mathbf{P}_{\mathbf{1}_{b_3}} \otimes \mathbf{P}_{\mathbf{1}_{b_1}} \otimes \mathbf{P}_{\mathbf{1}_{b_2}}, \quad \mathbf{P_1} = \mathbf{Q}_{\mathbf{1}_{b_3}} \otimes \mathbf{P}_{\mathbf{1}_{b_1}} \otimes \mathbf{P}_{\mathbf{1}_{b_2}}, \\ &\mathbf{P_2} = \mathbf{I}_{\mathbf{1}_{b_3}} \otimes \mathbf{Q}_{\mathbf{1}_{b_1}} \otimes \mathbf{P}_{\mathbf{1}_{b_2}}, \quad \mathbf{P_3} = \mathbf{I}_{\mathbf{1}_{b_3}} \otimes \mathbf{P}_{\mathbf{1}_{b_1}} \otimes \mathbf{Q}_{\mathbf{1}_{b_2}}, \\ &\mathbf{P_4} = \mathbf{I}_{\mathbf{1}_{b_3}} \otimes \mathbf{Q}_{\mathbf{1}_{b_1}} \otimes \mathbf{Q}_{\mathbf{1}_{b_2}}, \end{aligned} \tag{4}$$

and ξ_s, $s = 0, 1, 2, 3, 4$, denote the variance components

$$\xi_0 = \sigma_e^2, \;\; \xi_1 = \sigma_e^2 + b_1 b_2 \sigma_\gamma^2, \;\; \xi_2 = \sigma_e^2 + b_2 \sigma_\rho^2, \;\; \xi_3 = \sigma_e^2 + b_1 \sigma_\phi^2, \;\; \xi_4 = \sigma_e^2 + \sigma_\epsilon^2.$$

The ranks of matrices $\mathbf{P_s}$, $s = 0, 1, 2, 3, 4$, are respectively: 1, $b_3 - 1$, $b_3\,(b_1 - 1)$, $b_3\,(b_2 - 1)$, $b_3\,(b_1 - 1)\,(b_2 - 1)$ (see Łacka and Kozłowska, 2009).

The matrices are symmetric, idempotent and mutually orthogonal and sum up to the identity matrix.

Such a design has an orthogonal block structure (Houtman and Speed, 1983) and so the analysis of model (1) can be based on the method introduced by Nelder (1965a, 1965b).

Matrices (4) define orthogonal subspaces called *strata*. In the case of nested rows and columns they are, (see Bailey and Williams, 2007):

- stratum (0)
- stratum (1) between blocks
- stratum (2) between rows (within blocks)
- stratum (3) between columns (within blocks)
- stratum (4) between plots, also called rows-by-colums stratum or bottom stratum.

2. Estimation

2.1. Within stratum estimation

Let us note that the vector of observations (1) can be written as follows:

$$\mathbf{y} = \mathbf{y}_0 + \mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_3 + \mathbf{y}_4, \tag{5}$$

where

$$\mathbf{y}_s = \mathbf{P_s y} \tag{6}$$

belongs to stratum s, $s = 0, 1, 2, 3, 4$, and

$$\mathrm{E}(\mathbf{y}_s) = \mathbf{P_s}\boldsymbol{\Delta}'\boldsymbol{\tau} \quad \text{and} \quad \mathrm{Cov}(\mathbf{y}_s) = \xi_s \mathbf{P_s}.$$

Hence, the analysis based on model (1) can be split into stratum analyses based on models (6). Such models have been widely considered in the statistical literature, among others by Bailey (1981), Houtman and Speed (1983), Mejza and Mejza (1994), Caliński and Kageyama (2000, 2003, 2008), Kozłowska (2001), Bailey and Williams (2007), Łacka and Kozłowska (2009) and Łacka, Kozłowska and Kozłowski (2009). It is known that, due to the orthogonal block structure, the estimation of linear functions of treatment effects, $\mathbf{c}'\boldsymbol{\tau}$, can be based on the strata models (6) with the variance-covariance matrices $\mathrm{Cov}(\mathbf{y}_s) = \xi_s \mathbf{I}_n$, $s = 0, 1, 2, 3, 4$, that is, the Best Linear Unbiased Estimators of the functions, BLUE($\mathbf{c}'\boldsymbol{\tau}$), are the same in both the general and the simple linear model. In the case of block designs with nested rows and columns (NRC), the estimation of functions $\mathbf{c}'\boldsymbol{\tau}$ within the strata is based on the following sets of reduced normal equations:

$$\mathbf{C}_{(s)}\boldsymbol{\tau}_s = \mathbf{Q_s}, \quad s = 0, 1, 2, 3, 4,$$

where $\mathbf{C}_{(s)} = \boldsymbol{\Delta}\mathbf{P_s}\boldsymbol{\Delta'}$ and $\mathbf{Q_s} = \boldsymbol{\Delta}\mathbf{P_s}\mathbf{y}$. Let us note that $\mathbf{C}_{(s)}\mathbf{1} = 0$ for $s > 0$. Hence only treatment contrasts are estimable in the strata, i.e., the functions $\mathbf{c}'\boldsymbol{\tau}$ for which $\mathbf{c}'\mathbf{1} = 0$. Furthermore, for $s = 0$ the only estimable functions are $\mathbf{c}'\boldsymbol{\tau} = (\mathbf{c}'\mathbf{1}/n)\mathbf{r}'\boldsymbol{\tau}$, where $\mathbf{c}'\mathbf{1} \neq 0$ and $\mathbf{r}$ denotes the vector of replications. Hence, for the NRC designs, the statistical analysis of treatment contrasts can be done in the four strata ($s = 1, 2, 3, 4$). In this paper we consider connected designs only (i.e. $\mathrm{r}(\mathbf{C}_{(4)}) = v - 1$).

A treatment contrast $\mathbf{c}'\boldsymbol{\tau}$ is estimable in stratum s if

$$\mathbf{c}'\mathbf{C}^-_{(s)}\mathbf{C}_{(s)} = \mathbf{c}', \tag{7}$$

where $\mathbf{C}^-_{(s)}$ denotes a generalized inverse of $\mathbf{C}_{(s)}$, Rao and Mitra (1971). Then, the BLUE($\mathbf{c}'\boldsymbol{\tau}$) in stratum s is

$$(\widehat{\mathbf{c}'\boldsymbol{\tau}})_s = \mathbf{c}'\mathbf{C}^-_{(s)}\mathbf{Q_s},$$

and its variance is equal to

$$\mathrm{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}})_s = \xi_s \mathbf{c}'\mathbf{C}^-_{(s)}\mathbf{c}.$$

A design with orthogonal block structure defined by the projection operators $\mathbf{P_s}$ and with a treatment structure defined on the column space of $\boldsymbol{\Delta}$, $\mathcal{G} = \mathcal{R}(\boldsymbol{\Delta}')$, is *generally balanced* with respect to a decomposition $\mathcal{G} = \oplus_i \mathcal{G}_i$ if there exists a real matrix $\left\{\boldsymbol{\lambda}_{(s)i}\right\}$ such that for each s we have

$$\mathbf{G}\mathbf{P_s}\mathbf{G} = \textstyle\sum_i \boldsymbol{\lambda}_{(s)i}\mathbf{G}_i,$$

where $\mathbf{G}$ and $\mathbf{G}_i$ are the orthogonal projection operators from $\mathbb{R}^n$ to $\mathcal{G}$ and $\mathcal{G}_i$, respectively, and $\oplus$ denotes the direct sum.

Further in this paper we consider generally balanced NRC designs. This allows us to make inference on the same set of contrasts within several strata.

Note, that matrix $\mathbf{G}$ can be written as

$$\begin{aligned}\mathbf{G} = \;& \mathbf{\Delta}'(\mathbf{\Delta\Delta}')^{-1}\mathbf{\Delta} = \mathbf{\Delta}'\mathbf{R}^{-1}\mathbf{\Delta} = \mathbf{\Delta}'\mathbf{R}^{-1}\mathbf{\Delta\Delta}'\mathbf{R}^{-1}\mathbf{\Delta} = \\ = \;& \mathbf{\Delta}'\mathbf{R}^{-1}\mathbf{\Delta}\mathbf{I}\mathbf{\Delta}'\mathbf{R}^{-1}\mathbf{\Delta} = \mathbf{\Delta}'\mathbf{R}^{-1}\mathbf{\Delta}\left(\sum_{s=0}^{4}\mathbf{P_s}\right)\mathbf{\Delta}'\mathbf{R}^{-1}\mathbf{\Delta} = \\ = \;& \sum_{s=0}^{4}\mathbf{\Delta}'\mathbf{R}^{-1}\mathbf{\Delta}\mathbf{P_s}\mathbf{\Delta}'\mathbf{R}^{-1}\mathbf{\Delta} = \sum_{s=0}^{4}\mathbf{\Delta}'\mathbf{R}^{-1}\mathbf{C}_{(s)}\mathbf{R}^{-1}\mathbf{\Delta},\end{aligned}$$

where $\mathbf{R} = \mathbf{\Delta\Delta}'$ is a diagonal matrix with the numbers of treatment replications on the diagonal.

This means that the property of general balance is related to the spectral decomposition of matrices $\mathbf{\Delta}'\mathbf{R}^{-1}\mathbf{C}_{(s)}\mathbf{R}^{-1}\mathbf{\Delta}$ for $s = 0,1,2,3,4$. Hence, an NRC design is generally balanced if and only if these matrices are mutually commutative, that is

$$\begin{aligned}&\left(\mathbf{\Delta}'\mathbf{R}^{-1}\mathbf{C}_{(s)}\mathbf{R}^{-1}\mathbf{\Delta}\right)\left(\mathbf{\Delta}'\mathbf{R}^{-1}\mathbf{C}_{(s')}\mathbf{R}^{-1}\mathbf{\Delta}\right)\\ &= \left(\mathbf{\Delta}'\mathbf{R}^{-1}\mathbf{C}_{(s')}\mathbf{R}^{-1}\mathbf{\Delta}\right)\left(\mathbf{\Delta}'\mathbf{R}^{-1}\mathbf{C}_{(s)}\mathbf{R}^{-1}\mathbf{\Delta}\right)\end{aligned}$$

for $s, s' = 0,1,2,3,4$. By pre-multiplying the above by $\mathbf{\Delta}$ and post-multiplying by $\mathbf{\Delta}'$ we obtain a commutativity condition for matrices $\mathbf{R}^{-1/2}\mathbf{C}_{(s)}\mathbf{R}^{-1/2}$

$$\mathbf{C}_{(s)}\mathbf{R}^{-1}\mathbf{C}_{(s')} = \mathbf{C}_{(s')}\mathbf{R}^{-1}\mathbf{C}_{(s)} \quad \text{for} \quad s, s' = 0,1,2,3,4. \tag{8}$$

Note that, independently of the structure of matrices $\mathbf{P_s}$, matrix $\mathbf{R}^{-1/2}\mathbf{C}_{(0)}\mathbf{R}^{-1/2}$ commutes with all other matrices $\mathbf{R}^{-1/2}\mathbf{C}_{(s)}\mathbf{R}^{-1/2}$. Hence, condition (8) has to be met for $s, s' = 1,2,3,4$.

The eigenvalues $\lambda_{(s)i}$ of matrices $\mathbf{C}_{(s)}$ with respect to $\mathbf{R}$ $(0 \leqslant \lambda_{(s)i} \leqslant 1)$ are such that $\lambda_{(1)i} + \lambda_{(2)i} + \lambda_{(3)i} + \lambda_{(4)i} = 1$, $i \leqslant v$, and the respective eigenvectors $\mathbf{w}_i$ meet the conditions

$$\mathbf{C}_{(s)}\mathbf{w}_i = \lambda_{(s)i}\mathbf{R}\mathbf{w}_i, \quad s = 0,1,2,3,4 \quad i = 1,2,...,v.$$

The set of eigenvectors can be chosen to be $\mathbf{R}$–orthonormal, that is such that $\mathbf{w}_i'\mathbf{R}\mathbf{w}_i = 1$ and $\mathbf{w}_{i'}'\mathbf{R}\mathbf{w}_i = 0$ for $i \neq i'$, $i, i' = 1,2,...,v$. From the relation $\mathbf{C}_{(s)}\mathbf{1} = \mathbf{0}$ (for $s > 0$) it follows that at least one of the eigenvalues has to be equal to zero and the respective eigenvector can be chosen as $\mathbf{w}_v = n^{-1/2}\mathbf{1}$. The vectors $\mathbf{w}_i$ for $i < v$ constitute a basis for $\mathbf{c}_i$ which define treatment contrasts, further called the basic contrasts $\mathbf{c}_i'\boldsymbol{\tau}$ (Pearce, Caliński and Marshall, 1974), where

$$\mathbf{c}_i = \mathbf{R}\mathbf{w}_i \quad \text{for} \quad i = 1,2,...,v-1. \tag{9}$$

The basic contrasts are such that $\mathbf{c}_i'\mathbf{R}^{-1}\mathbf{c}_i = 1$, $\mathbf{c}_i'\mathbf{R}^{-1}\mathbf{c}_{i'} = 0$ for $i \neq i'$, $i, i' = 1, 2, ..., v$ and $\mathbf{c}_i'\mathbf{1} = 0$ for $i < v$.

Note that $\boldsymbol{\Delta}'\mathbf{R}^{-1}\mathbf{C}_{(s)}\mathbf{R}^{-1}\boldsymbol{\Delta} = \sum_{i=1}^{v} \lambda_{(s)i}\boldsymbol{\Delta}'\mathbf{w}_i\mathbf{w}_i'\boldsymbol{\Delta} = \sum_{i=1}^{v} \lambda_{(s)i}\mathbf{G}_i$, where $\mathbf{G}_i = \boldsymbol{\Delta}'\mathbf{w}_i\mathbf{w}_i'\boldsymbol{\Delta}$ are orthogonal projection operators. Hence, the NRC designs have the property of general balance with respect to decomposition $\mathcal{G} = \mathcal{R}(\boldsymbol{\Delta}') = \mathcal{R}(\boldsymbol{\Delta}'\mathbf{w}_1) \oplus \mathcal{R}(\boldsymbol{\Delta}'\mathbf{w}_2) \oplus ... \oplus \mathcal{R}(\boldsymbol{\Delta}'\mathbf{w}_v)$, that is, with respect to the functions $\mathbf{c}_i'\boldsymbol{\tau}$. The property of general balance is related to the subspaces generated by the eigenvectors of the matrices $\mathbf{C}_{(s)}$.

In the s stratum, the BLUE($\mathbf{c}_i'\boldsymbol{\tau}$) is

$$(\widehat{\mathbf{c}_i'\boldsymbol{\tau}})_s = \lambda_{(s)i}^{-1}\mathbf{w}_i'\mathbf{Q_s} \quad \text{for} \quad s = 1, 2, 3, 4, \; i = 1, 2, ..., v - 1, \tag{10}$$

and its variance is given by $\mathrm{Var}(\widehat{\mathbf{c}_i'\boldsymbol{\tau}})_s = \xi_s\lambda_{(s)i}^{-1}$. Hence, in the s stratum, the only estimable contrasts are those related to the non-zero eigenvalues $\lambda_{(s)i}$ (see also Houtman and Speed, 1983). The eigenvalue $\lambda_{(s)i}$ related to the basic contrast $\mathbf{c}_i'\boldsymbol{\tau}$ is called the *efficiency factor of estimation* of the contrast in stratum s.

2.2. Combined estimators – known stratum variances

It has been shown in the statistical literature that if a treatment contrast is estimable in the bottom stratum only, then the stratum BLUE of the contrast is also the BLUE in the full model (see Bailey, 1981; Mejza, 1992). Many contrasts, however, are estimable in more than one stratum. The efficiency of estimation ith basic contrast in each stratum is given by the eigenvalue $\lambda_{(s)i}$ and it is not obvious how to use the information which is split among the strata. In this section we assume that the stratum variances are known and we show some properties of combined estimators.

The variance-covariance matrix (2) can be written in the following form

$$\begin{aligned} \mathrm{Cov}(\mathbf{y}) = \; & (\sigma_\epsilon^2 + \sigma_e^2)\left(\mathbf{D}'\boldsymbol{\Gamma}_1\mathbf{D} + \mathbf{D}_1'\boldsymbol{\Gamma}_2\mathbf{D}_1 + \mathbf{D}_2'\boldsymbol{\Gamma}_3\mathbf{D}_2 + \mathbf{I}\right) \\ & = \xi_4\left(\mathbf{D}'\boldsymbol{\Gamma}_1\mathbf{D} + \mathbf{D}_1'\boldsymbol{\Gamma}_2\mathbf{D}_1 + \mathbf{D}_2'\boldsymbol{\Gamma}_3\mathbf{D}_2 + \mathbf{I}\right), \end{aligned} \tag{11}$$

where

$$\begin{aligned} \boldsymbol{\Gamma}_1 = \; & \left(\frac{\sigma_\gamma^2 + \sigma_\epsilon^2/(b_1b_2) - \sigma_\rho^2/b_1 - \sigma_\phi^2/b_2}{\sigma_\epsilon^2 + \sigma_e^2}\right)\mathbf{I} - \left(\frac{\sigma_\gamma^2/b_3}{\sigma_\epsilon^2 + \sigma_e^2}\right)\mathbf{11}' = \gamma_1\mathbf{I} - \left(\frac{\sigma_\gamma^2/b_3}{\sigma_\epsilon^2 + \sigma_e^2}\right)\mathbf{11}', \\ \boldsymbol{\Gamma}_2 = \; & \left(\frac{\sigma_\rho^2 - \sigma_\epsilon^2/b_2}{\sigma_\epsilon^2 + \sigma_e^2}\right)\mathbf{I} = \gamma_2\mathbf{I}, \\ \boldsymbol{\Gamma}_3 = \; & \left(\frac{\sigma_\phi^2 - \sigma_\epsilon^2/b_1}{\sigma_\epsilon^2 + \sigma_e^2}\right)\mathbf{I} = \gamma_3\mathbf{I}, \end{aligned} \tag{12}$$

and

$$
\begin{aligned}
\gamma_1 &= \tfrac{(\xi_1-\xi_2-\xi_3+\xi_4)/b_1b_2}{\xi_4},\\
\gamma_2 &= \tfrac{(\xi_2-\xi_4)/b_2}{\xi_4},\\
\gamma_3 &= \tfrac{(\xi_3-\xi_4)/b_1}{\xi_4}.
\end{aligned}
\tag{13}
$$

Assuming that the variance components in (2) or that the parameters (13) are known, it can be shown, analogously to Lemma 5.5.1 of Caliński and Kageyama (2000), that the following results hold.

Lemma 1. For the model (1), with the expectation vector $\mathrm{E}(\mathbf{y}) = \boldsymbol{\Delta}'\boldsymbol{\tau}^\circ$, where $\boldsymbol{\tau}^\circ = \boldsymbol{\tau} + \mathbf{1}\mu$, and covariance matrix of the form (2) or, equivalently, (11) with known true values of γ_i, $i = 1, 2, 3$, we have that:

1. Any function $\mathbf{w}'\mathbf{y}$ that is BLUE of its expectation $\mathbf{w}'\boldsymbol{\Delta}'\boldsymbol{\tau}^\circ$,

2. A vector that is the BLUE of $\mathrm{E}(\mathbf{y}) = \boldsymbol{\Delta}'\boldsymbol{\tau}^\circ$,

3. A vector that gives the residuals,

all remain unchanged when deleting the term $\left(\frac{\sigma_\gamma^2/b_3}{\sigma_\epsilon^2+\sigma_e^2}\right)\mathbf{11}'$ in (12), i.e., by reducing the covariance matrix (2) to $\mathrm{Cov}(\mathbf{y}) = \xi_4\mathbf{T}$, where $\mathbf{T} = \gamma_1\mathbf{D}'\mathbf{D} + \gamma_2\mathbf{D}_1'\mathbf{D}_1 + \gamma_3\mathbf{D}_2'\mathbf{D}_2 + \mathbf{I}$.

This together with Theorem 3.2(c) of (Rao, 1974) gives the BLUE$(\boldsymbol{\Delta}'\boldsymbol{\tau}^\circ)$ in the form

$$\widehat{\boldsymbol{\Delta}'\boldsymbol{\tau}^\circ} = \boldsymbol{\Delta}'\left(\boldsymbol{\Delta}\mathbf{T}^{-1}\boldsymbol{\Delta}'\right)^{-1}\boldsymbol{\Delta}\mathbf{T}^{-1}\mathbf{y},$$

where $\mathbf{T}^{-1} = \mathbf{P}_4 + \frac{\xi_4}{\xi_3}\mathbf{P}_3 + \frac{\xi_4}{\xi_2}\mathbf{P}_2 + \frac{\xi_4}{\xi_1}(\mathbf{P}_1 + \mathbf{P}_0)$. Hence, we obtain the following theorem.

Theorem 1. Under the assumptions of Lemma 1 we have:

1. The BLUE of $\boldsymbol{\tau}^\circ$ is of the form

$$\widehat{\boldsymbol{\tau}^\circ} = \left(\boldsymbol{\Delta}\mathbf{T}^{-1}\boldsymbol{\Delta}'\right)^{-1}\boldsymbol{\Delta}\mathbf{T}^{-1}\mathbf{y} = \mathbf{C}_c^{-1}\mathbf{Q}_c, \tag{14}$$

where $\mathbf{C}_c = \boldsymbol{\Delta}\mathbf{T}^{-1}\boldsymbol{\Delta}'$ and $\mathbf{Q}_c = \boldsymbol{\Delta}\mathbf{T}^{-1}\mathbf{y}$.

2. The covariance matrix of $\widehat{\boldsymbol{\tau}^\circ}$ is

$$\mathrm{Cov}(\widehat{\boldsymbol{\tau}^\circ}) = \xi_4 \left(\boldsymbol{\Delta}\mathbf{T}^{-1}\boldsymbol{\Delta}'\right)^{-1} - \left(\frac{\sigma_\gamma^2/b_3}{\xi_4}\right)\mathbf{1}\mathbf{1}'.$$

3. The BLUE of $\mathbf{c}'\boldsymbol{\tau}^\circ$ for any $\mathbf{c}$ is $\mathbf{c}'\widehat{\boldsymbol{\tau}^\circ}$, with the variance $\mathbf{c}'\mathrm{Cov}(\widehat{\boldsymbol{\tau}^\circ})\mathbf{c}$, which reduces to

$$\mathrm{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}^\circ}) = \xi_4\mathbf{c}'\left(\boldsymbol{\Delta}\mathbf{T}^{-1}\boldsymbol{\Delta}'\right)^{-1}\mathbf{c}. \tag{15}$$

4. The MINIQUE of ξ_4 is

$$\widehat{\xi_4} = \mathbf{y}'\mathbf{K}\mathbf{y}/(n-v),$$

where $\mathbf{K} = \mathbf{T}^{-1} - \mathbf{T}^{-1}\boldsymbol{\Delta}'\left(\boldsymbol{\Delta}\mathbf{T}^{-1}\boldsymbol{\Delta}'\right)^{-1}\boldsymbol{\Delta}\mathbf{T}^{-1}$.

For each basic contrast $\mathbf{c}_i'\boldsymbol{\tau}^\circ = \mathbf{c}_i'\boldsymbol{\tau}$. For $\mathbf{c}_i$ of the form (9) we have

$$\widehat{\mathbf{c}_i'\boldsymbol{\tau}^\circ} = \widehat{\mathbf{c}_i'\boldsymbol{\tau}} = \sum_{s=1}^{4} u_{(s)i}\left(\widehat{\mathbf{c}_i'\boldsymbol{\tau}}\right)_s, \tag{16}$$

where

$$u_{(s)i} = \frac{\lambda_{(s)i}}{\frac{\xi_s}{\xi_1}\lambda_{(1)i} + \frac{\xi_s}{\xi_2}\lambda_{(2)i} + \frac{\xi_s}{\xi_3}\lambda_{(3)i} + \frac{\xi_s}{\xi_4}\lambda_{(4)i}} \quad \text{for} \quad s = 1,2,3,4, \tag{17}$$

(see also Mejza and Mejza, 1994). Note also that

$$\begin{aligned} \mathbf{Q}_c = {} & \boldsymbol{\Delta}\mathbf{T}^{-1}\mathbf{y} = \mathbf{Q_4} + \tfrac{\xi_4}{\xi_3}\mathbf{Q_3} + \tfrac{\xi_4}{\xi_2}\mathbf{Q_2} + \tfrac{\xi_4}{\xi_1}\left(\mathbf{Q_1} + \mathbf{Q_0}\right), \\ \mathbf{C}_c = {} & \boldsymbol{\Delta}\mathbf{T}^{-1}\boldsymbol{\Delta}' = \mathbf{C}_{(4)} + \tfrac{\xi_4}{\xi_3}\mathbf{C}_{(3)} + \tfrac{\xi_4}{\xi_2}\mathbf{C}_{(2)} + \tfrac{\xi_4}{\xi_1}\left(\mathbf{C}_{(1)} + \mathbf{C}_{(0)}\right). \end{aligned}$$

2.3. Combined estimators – unknown stratum variances

In the previous section we assumed that the variance components were known. This is useful for looking into some properties of the combined estimators and to see what is the "ideal" solution. However, in most practical situations, the assumption will not be met and so we need to consider methods of combining the information from different strata when using estimators of the unknown variance components. In the statistical literature several techniques for combining the estimators have been suggested. A comparison of some of the methods is given by Caliński and Kageyama (2000). Here we present three approaches to the estimation and combination of the stratum information.

Let us write the residual sum of squares in the full model as:

$$\begin{aligned}\left\|\left(\mathbf{I}_n - \mathbf{P}_{\boldsymbol{\Delta}'(\mathbf{T}^{-1})}\right)\mathbf{y}\right\|^2_{\mathbf{T}^{-1}} &= \mathbf{y}'\mathbf{KTKy} = \mathbf{y}'\mathbf{Ky}\\ &= \gamma_1\mathbf{y}'(\mathbf{KD}'\mathbf{DK})\mathbf{y} + \gamma_2\mathbf{y}'(\mathbf{KD}_1'\mathbf{D}_1\mathbf{K})\mathbf{y}\\ &\quad +\gamma_3\mathbf{y}'(\mathbf{KD}_2'\mathbf{D}_2\mathbf{K})\mathbf{y} + \mathbf{y}'\mathbf{KKy}.\end{aligned}$$

Following the approach used by Nelder (1968) and generalized by Caliński and Kageyama (1996), the simultaneous estimators of ξ_4, γ_1, γ_2 and γ_3 can be obtained by equating the partial sums of squares in the equation above to their expectations:

$$\begin{aligned}\mathrm{E}\left(\mathbf{y}'(\mathbf{KD}'\mathbf{DK})\mathbf{y}\right) &= \xi_4\ \mathrm{tr}(\mathbf{KD}'\mathbf{DKT}) = \xi_4\ \mathrm{tr}(\mathbf{KD}'\mathbf{D}),\\ \mathrm{E}\left(\mathbf{y}'(\mathbf{KD}_1'\mathbf{D}_1\mathbf{K})\mathbf{y}\right) &= \xi_4\ \mathrm{tr}(\mathbf{KD}_1'\mathbf{D}_1\mathbf{KT}) = \xi_4\ \mathrm{tr}(\mathbf{KD}_1'\mathbf{D}_1),\\ \mathrm{E}\left(\mathbf{y}'(\mathbf{KD}_2'\mathbf{D}_2\mathbf{K})\mathbf{y}\right) &= \xi_4\ \mathrm{tr}(\mathbf{KD}_2'\mathbf{D}_2\mathbf{KT}) = \xi_4\ \mathrm{tr}(\mathbf{KD}_2'\mathbf{D}_2),\\ \mathrm{E}\left(\mathbf{y}'\mathbf{KKy}\right) &= \xi_4\ \mathrm{tr}(\mathbf{KKT}) = \xi_4\ \mathrm{tr}(\mathbf{K}).\end{aligned}$$

This can be written as:

$$\begin{bmatrix}\mathbf{y}'(\mathbf{KD}'\mathbf{DK})\mathbf{y}\\ \mathbf{y}'(\mathbf{KD}_1'\mathbf{D}_1\mathbf{K})\mathbf{y}\\ \mathbf{y}'(\mathbf{KD}_2'\mathbf{D}_2\mathbf{K})\mathbf{y}\\ \mathbf{y}'\mathbf{KKy}\end{bmatrix} = \mathbf{X}\begin{bmatrix}\xi_4\gamma_1\\ \xi_4\gamma_2\\ \xi_4\gamma_3\\ \xi_4\end{bmatrix}, \tag{18}$$

where:

$$\mathbf{X} = \begin{bmatrix}\mathrm{tr}(\mathbf{KD}'\mathbf{DKD}'\mathbf{D}) & \mathrm{tr}(\mathbf{KD}'\mathbf{DKD}_1'\mathbf{D}_1) & \mathrm{tr}(\mathbf{KD}'\mathbf{DKD}_2'\mathbf{D}_2) & \mathrm{tr}(\mathbf{KD}'\mathbf{DK})\\ \mathrm{tr}(\mathbf{KD}_1'\mathbf{D}_1\mathbf{KD}'\mathbf{D}) & \mathrm{tr}(\mathbf{KD}_1'\mathbf{D}_1\mathbf{KD}_1'\mathbf{D}_1) & \mathrm{tr}(\mathbf{KD}_1'\mathbf{D}_1\mathbf{KD}_2'\mathbf{D}_2) & \mathrm{tr}(\mathbf{KD}_1'\mathbf{D}_1\mathbf{K})\\ \mathrm{tr}(\mathbf{KD}_2'\mathbf{D}_2\mathbf{KD}'\mathbf{D}) & \mathrm{tr}(\mathbf{KD}_2'\mathbf{D}_2\mathbf{KD}_1'\mathbf{D}_1) & \mathrm{tr}(\mathbf{KD}_2'\mathbf{D}_2\mathbf{KD}_2'\mathbf{D}_2) & \mathrm{tr}(\mathbf{KD}_2'\mathbf{D}_2\mathbf{K})\\ \mathrm{tr}(\mathbf{KKD}'\mathbf{D}) & \mathrm{tr}(\mathbf{KKD}_1'\mathbf{D}_1) & \mathrm{tr}(\mathbf{KKD}_2'\mathbf{D}_2) & \mathrm{tr}(\mathbf{KK})\end{bmatrix}.$$

The equations (18) have no analytical solution, as both the coefficient matrix on the right-hand side and the vector of the quadratic forms in $\mathbf{y}$ on the left hand side contain the unknown parameters ξ_4, γ_1, γ_2 and γ_3. Therefore, the equations have to be solved numerically by an iterative procedure.

Now, by inserting matrix $\widehat{\mathbf{T}}^{-1}$, obtained by replacing γ_1, γ_2 and γ_3 by their estimators $\widehat{\gamma}_1$, $\widehat{\gamma}_2$ and $\widehat{\gamma}_3$, in place of $\mathbf{T}^{-1}$ in formula (14), and by using the relations:

$$\begin{aligned} \tfrac{\xi_1}{\xi_4} &= b_1 b_2 \gamma_1 + b_2\gamma_2 + b_1\gamma_3 + 1, \\ \tfrac{\xi_2}{\xi_4} &= b_2\gamma_2 + 1, \\ \tfrac{\xi_3}{\xi_4} &= b_1\gamma_3 + 1, \end{aligned}$$

we get an empirical estimator of the form

$$\widetilde{\boldsymbol{\tau}}^{\circ} = \left(\boldsymbol{\Delta}\widehat{\mathbf{T}}^{-1}\boldsymbol{\Delta}'\right)^{-1}\boldsymbol{\Delta}\widehat{\mathbf{T}}^{-1}\mathbf{y} = \widehat{\mathbf{C}}_c^{-1}\widehat{\mathbf{Q}}_c,$$

where $\widehat{\mathbf{C}}_c = \boldsymbol{\Delta}\widehat{\mathbf{T}}^{-1}\boldsymbol{\Delta}'$ and $\widehat{\mathbf{Q}}_c = \boldsymbol{\Delta}\widehat{\mathbf{T}}^{-1}\mathbf{y}$. Then, the weights (17) can be replaced by their estimators $\widehat{u}_{(s)i}$.

To solve the problem of estimation in model (5) we propose estimating ξ_s, $s = 1, 2, 3, 4$, based on the following decomposition of vector $\mathbf{y}$,

$$\mathbf{y} = \mathbf{P}_{\boldsymbol{\Delta}'\mathbf{V}^{-1}}\mathbf{y} + (\mathbf{I} - \mathbf{P}_{\boldsymbol{\Delta}'\mathbf{V}^{-1}})\mathbf{y},$$

where $\mathbf{V} = \mathrm{Cov}(\mathbf{y})$ and $\mathbf{P}_{\boldsymbol{\Delta}'\mathbf{V}^{-1}} = \boldsymbol{\Delta}'(\boldsymbol{\Delta}\mathbf{V}^{-1}\boldsymbol{\Delta}')^{-1}\boldsymbol{\Delta}\mathbf{V}^{-1}$. With this decomposition, as suggested by Nelder (1968), it is natural to consider the solutions of the following equations as the estimators of ξ_s,

$$\left\{\begin{aligned} ||\mathbf{P_1}(\mathbf{I} - \mathbf{P}_{\boldsymbol{\Delta}'\mathbf{V}^{-1}})\mathbf{y}||^2 &= \xi_1 d_1, \\ ||\mathbf{P_2}(\mathbf{I} - \mathbf{P}_{\boldsymbol{\Delta}'\mathbf{V}^{-1}})\mathbf{y}||^2 &= \xi_2 d_2, \\ ||\mathbf{P_3}(\mathbf{I} - \mathbf{P}_{\boldsymbol{\Delta}'\mathbf{V}^{-1}})\mathbf{y}||^2 &= \xi_3 d_3, \\ ||\mathbf{P_4}(\mathbf{I} - \mathbf{P}_{\boldsymbol{\Delta}'\mathbf{V}^{-1}})\mathbf{y}||^2 &= \xi_4 d_4, \end{aligned}\right. \tag{19}$$

where $d_s = \mathrm{tr}\{\mathbf{P_s}(\mathbf{I} - \mathbf{P}_{\boldsymbol{\Delta}'\mathbf{V}^{-1}})\}$. In this case also, there are no analytical solutions of (19) and Nelder (1968) suggested using an iterative procedure to obtain some approximations to the unavailable BLUEs. The estimators obtained by solving (19) are equivalent to those obtained from (18).

On the other hand, the stratum estimators of the treatment contrasts (10) are mutually independent and also independent of the error sums of squares (see Searle, 1971; Ambroży and Mejza, 2006; Łacka, 2009). This property allows us to combine the information from various strata. We suggest use of the method of combining stratum estimators described by Bhattacharya (1978, 1979), and Shinozaki (1978), among others. If a contrast $\mathbf{c}_i'\boldsymbol{\tau}$ is estimable in the four strata (or in less then four) then the combined estimator obtained by this method is given by

$$\widehat{\mathbf{c}_i'\boldsymbol{\tau}} = \frac{\sum\limits_{s=1}^{4} f_s \widehat{\xi}_s^{-1} (\widehat{\mathbf{c}_i'\boldsymbol{\tau}})_s}{\sum\limits_{s=1}^{4} f_s \widehat{\xi}_s^{-1}} \tag{20}$$

where f_s are arbitrary constants. This is a uniformly better estimator than any stratum estimator $(\widehat{\mathbf{c}_i'\boldsymbol{\tau}})_s$ if and only if the constants f_s are such that for all $s \neq s'$, $s, s' = 1, 2, 3, 4$, the following inequality holds

$$\frac{f_{s'}}{f_s} \leqslant \frac{2v_{E(s)}(v_{E(s')} - 4)}{v_{E(s')}(v_{E(s)} + 2)}, \quad \text{where} \quad v_{E(s)} = r(\mathbf{P_s}) - r(\mathbf{C}_{(s)}). \tag{21}$$

According to Shinozaki (1978), the constants f_s and $f_{s'}$ can be chosen so that the above inequality is met only if for all $s \neq s'$, $s, s' = 1, 2, 3, 4$, $v_{E(s)} \geqslant 7$ and $(v_{E(s)} - 6)(v_{E(s')} - 6) \geqslant 16$. Then Shinozaki (1978) suggests taking $f_s = (v_{E(s)} - 2)/v_{E(s)}$.

Another way of calculating the combined estimator is to take $f_s = \lambda_{(s)i}$ and $\widehat{\xi}_s = v_{E(s)}^{-1} SS_{E(s)}$, where $SS_{E(s)}$ denotes the residual sum of squares in stratum s. Then, (20) becomes

$$\widehat{\mathbf{c}_i'\boldsymbol{\tau}} = \frac{\sum\limits_{s=1}^{4} v_{E(s)} (SS_{E(s)})^{-1} \mathbf{w}_i' \mathbf{Q_s}}{\sum\limits_{s=1}^{4} \lambda_{(s)i} v_{E(s)} (SS_{E(s)})^{-1}},$$

which is uniformly better than the stratum estimators (10) if and only if the condition (21) holds for the chosen efficiencies. This is also an empirical estimator, that is the unknown variance components are replaced by their estimates. Good statistical properties of the empirical estimators hold only for large sample sizes.

3. Hypothesis testing for NRC designs

3.1. Testing within strata

In the analysis of NRC designs, a researcher may be interested not only in estimation of treatment parameters and their linear functions, particularly contrasts, but also in testing hypotheses concerning such functions.

In this section we present the analysis of variance in stratum s, including tests for a general hypothesis as well as for some specific hypotheses of non-significance of the estimable contrasts in stratum s. The tests are based on the orthogonal decomposition of the variance-covariance matrix (3) and of the column space of $\mathbf{\Delta}'$ as defined in Section 2.1. Further, we assume that the random effects of model (1) are normally distributed, as follows

$$\begin{aligned}
&\boldsymbol{\gamma} \sim N(\mathbf{0}, \sigma_\gamma^2 \mathbf{Q}_{\mathbf{1}_{b_3}}),\\
&\boldsymbol{\rho} \sim N(\mathbf{0}, \sigma_\rho^2 (\mathbf{I}_{b_3} \otimes \mathbf{Q}_{\mathbf{1}_{b_1}})),\\
&\boldsymbol{\phi} \sim N(\mathbf{0}, \sigma_\phi^2 (\mathbf{I}_{b_3} \otimes \mathbf{Q}_{\mathbf{1}_{b_2}})),\\
&\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma_\epsilon^2 (\mathbf{I}_{b_3} \otimes \mathbf{Q}_{\mathbf{1}_{b_1}} \otimes \mathbf{Q}_{\mathbf{1}_{b_2}})),\\
&\mathbf{e} \sim N(\mathbf{0}, \sigma_e^2 \mathbf{I}).
\end{aligned}$$

The analysis of variance in stratum s is based on splitting the sums of squares

$$\begin{aligned}
\mathbf{y}'\mathbf{P_s}\mathbf{y} = {} & \mathbf{y}'\left(\mathbf{P_s} + \mathbf{P_s}\mathbf{\Delta}'\mathbf{C}_{(s)}^{-}\mathbf{\Delta}\mathbf{P_s} - \mathbf{P_s}\mathbf{\Delta}'\mathbf{C}_{(s)}^{-}\mathbf{\Delta}\mathbf{P_s}\right)\mathbf{y} = \\
= {} & \mathbf{y}'\left(\mathbf{P_s}\mathbf{\Delta}'\mathbf{C}_{(s)}^{-}\mathbf{\Delta}\mathbf{P_s}\right)\mathbf{y} + \mathbf{y}'\left(\mathbf{P_s} - \mathbf{P_s}\mathbf{\Delta}'\mathbf{C}_{(s)}^{-}\mathbf{\Delta}\mathbf{P_s}\right)\mathbf{y},
\end{aligned}$$

that is,

$$SS_{G(s)} = SS_{T(s)} + SS_{E(s)} \quad \text{for} \quad s = 1, 2, 3, 4.$$

Here $SS_{G(s)} = \mathbf{y}'\mathbf{P_s}\mathbf{y}$ denotes the stratum total sum of squares, $SS_{T(s)} = \mathbf{y}'\left(\mathbf{P_s}\mathbf{\Delta}'\mathbf{C}_{(s)}^{-}\mathbf{\Delta}\mathbf{P_s}\right)\mathbf{y} = \mathbf{Q_s}'\mathbf{C}_{(s)}^{-}\mathbf{Q_s}$ denotes the stratum sum of squares for treatments and $SS_{E(s)} = SS_{G(s)} - SS_{T(s)} = \mathbf{y}'\left(\mathbf{P_s} - \mathbf{P_s}\mathbf{\Delta}'\mathbf{C}_{(s)}^{-}\mathbf{\Delta}\mathbf{P_s}\right)\mathbf{y}$ is the stratum residual sum of squares. The unbiased estimator of the stratum variance is the mean square error of that stratum (see Searle, 1971; Łacka, 2009), that is

$$\widehat{\xi}_s = v_{E(s)}^{-1} SS_{E(s)}, \quad \text{for} \quad s = 1, 2, 3, 4. \tag{22}$$

Due to the treatment structure described in Section 2.1, the sum of squares for treatments in stratum s can be written as the following combination of treatment contrasts

$$SS_{T(s)} = \sum_{i:\lambda_{(s)i} \neq 0} \lambda_{(s)i} \left[(\widehat{\mathbf{c}_i'\boldsymbol{\tau}})_s\right]^2 = \sum_{i:\lambda_{(s)i} \neq 0} \lambda_{(s)i}^{-1} \left(\mathbf{w}_i'\mathbf{Q_s}\right)^2, \quad \text{for} \quad s = 1, 2, 3, 4.$$

By the assumptions of the normality of the observations, all the mean squares in stratum s divided by respective variances follow the chi-square distribution with respective degrees of freedom. Also, the mean squares for treatments and for error are independent (see Searle, 1971; Łacka, 2009). The stratum general hypothesis we are interested in can be stated as follows

$$\mathrm{H}_{0(s)} : \mathbf{C}_{(s)}\boldsymbol{\tau} = \mathbf{0} \quad \text{where} \quad s = 1, 2, 3, 4. \tag{23}$$

This can be equivalently written as $\mathrm{H}_{0(s)} : \boldsymbol{\tau}'\mathbf{C}_{(s)}\boldsymbol{\tau} = 0$. This hypothesis states that all the basic contrasts estimable in stratum s are equal to zero. Under this hypothesis, the test function

$$F_{0(s)} = \frac{v_{T(s)}^{-1} SS_{T(s)}}{v_{E(s)}^{-1} SS_{E(s)}}, \quad \text{for} \quad s = 1, 2, 3, 4$$

has the central F distribution with $v_{T(s)} = r(\mathbf{C}_{(s)})$ and $v_{E(s)}$ degrees of freedom.

The null hypothesis (23) is expressed in terms of the stratum information matrix $\mathbf{C}_{(s)}$, hence it depends on the design. It may be more interesting for an experimenter to test non-significance of a specific basic contrast $\mathbf{c}_i'\boldsymbol{\tau}$ estimable in stratum s, that is

$$\mathrm{H}_{0(s)i} : (\mathbf{c}_i'\boldsymbol{\tau})_s = 0 \quad \text{where} \quad s = 1, 2, 3, 4,\ i = 1, ..., v-1. \tag{24}$$

The test function for this hypothesis is given by

$$F_{0(s)i} = \frac{\lambda_{(s)i}^{-1} \left(\mathbf{w}_i'\mathbf{Q_s}\right)^2}{v_{E(s)}^{-1} SS_{E(s)}},$$

which under the null hypothesis (24) has the central F distribution with 1 and $v_{E(s)}$ degrees of freedom.

3.2. Combined tests

It is known that, if the rejection of the overall null hypothesis H_0 is implied by the rejection of one of the four stratum hypothesis $\mathrm{H}_{0(s)}$, $s = 1, 2, 3, 4$, then all hypotheses concerning basic contrasts, $\mathrm{H}_{0(s)i}$, should be tested in this stratum. Otherwise, a combined test should be used to verify significance of the basic contrasts (this problem was considered for example for block design in 1985 by Mejza). In this section we present a combined test, which in the case of unknown variance components has an approximate F distribution.

First, let us assume that the variance components are known. Then, to apply exact F tests of the hypotheses

$$\mathrm{H}_{0i} : \mathbf{c}_i'\boldsymbol{\tau} = 0 \tag{25}$$

the BLUEs of $\mathbf{c}_i'\boldsymbol{\tau}$ together with their variances are needed. The BLUE of the basic contrast $\mathbf{c}_i'\boldsymbol{\tau}$ is the combined estimator of the form (16) with variance given by (15) for $i = 1, ..., v-1$. It is known that the squared combined estimator divided by its variance has a χ^2 distribution with 1 degree of freedom. If in this statistic the variance component ξ_4 is replaced by its estimate (22), then the obtained test function has an approximately F distribution with 1 and $r(\mathbf{P_4}) - r(\mathbf{C}_{(4)})$ degrees of freedom.

In practice the γ values (13) need to be estimated. Here we may use the combined empirical estimator to obtain the test function

$$F = \frac{\widehat{u}_{(4)i}^{-1}\lambda_{(4)i}(\widetilde{\mathbf{c}_i'\boldsymbol{\tau}})^2}{\mathbf{y}'\left(\mathbf{P_4} - \mathbf{P_4}\boldsymbol{\Delta}'\mathbf{C}_{(4)}^{-}\boldsymbol{\Delta}\mathbf{P_4}\right)\mathbf{y}}$$

which, under the null hypothesis, has an approximate F distribution.

Another test of the hypothesis H_{0i}, suggested by Fisher (1954), uses the statistic

$$\chi^2 = -2\ln \alpha_{1i}\alpha_{2i}\alpha_{3i}\alpha_{4i}, \tag{26}$$

where

$$\alpha_{si} = \mathrm{P}\left(F_{1,v_{E(s)}} > F_{0(s)i}\,\middle|\,\mathrm{H}_{0(s)i}\right), \quad \text{for} \quad s = 1, 2, 3, 4.$$

Under the null hypothesis (25) the test function (26) has an approximate chi-square distribution with eight degrees of freedom. The number of degrees of freedom is equal to twice the number of strata used in the recovery of information (estimation). Hence, when all four strata are included we have eight degrees of freedom. Obviously, we include only the strata where the tested contrast is estimable.

4. Concluding remarks

In this paper we have considered designs for which the general hypothesis can be tested in at least one stratum. If it can be tested in one stratum only, then because we consider only the connected NRC designs, it is the bottom one. Furthermore all the basic contrasts are estimable in this stratum. If a treatment contrast is estimable in more than one stratum then, to improve the inference, we can either use combined estimators and tests, or use NRC designs for which the most interesting contrasts are estimated with the highest possible efficiency in the bottom stratum. We may then be able to restrict the inference to this stratum.

The stratum variances of the basic contrasts are inversely proportional to the respective eigenvalues of the stratum information matrix with respect to matrix **R**. That is, the higher is the efficiency the smaller is the variance of the contrast. The stratum efficiency factors are also related to various optimality properties of the designs. Hence, the choice of designs with specific patterns of the eigenvalues may further improve inference about the contrasts.

Acknowledgments

This research was supported by the British-Polish Young Scientists Programme, grant WAR/342/116.

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